Based on K. H. Rosen: Discrete Mathematics and its Applications.

Lecture 5: Nested quantifiers. Section 1.5

1 Nested quantifiers

Nested quantifiers are quantifiers that occur within the scope of other quantifiers. **Quantifiers order matters!**

Example 1. Let $Q(x,y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists\, y \,\forall\, x\, Q(x,y)$ and $\forall\, y \,\exists\, x\, Q(x,y)$, where the domain for all variables consists of all real numbers?
Ans: The quantification

$$\exists\, y \,\forall\, x\, Q(x,y)$$

denotes the proposition "There is a real number y such that for every real number x, we have $Q(x,y)$."
No matter what value of y is chosen, there is only one value of x for which $x+y=0$. Because there is no real number y such that $x+y=0$ for all real numbers x, the statement $\exists\, y \forall\, x\, Q(x,y)$ is false. On the other hand, the quantification

$$\forall\, y \,\exists\, x\, Q(x,y)$$

denotes the proposition "For every real number x there is a real number y such that we have $Q(x,y)$." Given a real number x, there is a real number y such that $x+y=0$; namely, $y=-x$. Hence, the statement $\forall\, y \,\exists\, x\, Q(x,y)$ is true.

Observe however that the order of **nested universal quantifiers** in a statement without other quantifiers can be changed without changing the meaning of the quantified statement:

$$\forall\, x \,\forall\, y\, P(x,y) \quad \text{have the same meaning as} \quad \forall\, y \,\forall\, x\, P(x,y).$$

Example 2. Translate the statement "Every real number except zero has a multiplicative inverse." (A multiplicative inverse of a real number x is a real number y such that $xy = 1$.)

$$\forall\, x\, ((x \neq 0) \to \exists\, y\, xy = 1).$$

Example 3. Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.
Ans: The statement "If a person is female and is a parent, then this person is someone's mother" can be expressed as "For every person x, if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y." We introduce the propositional functions $F(x)$ to represent "x is female,"

$P(x)$ to represent "x is a parent," and $M(x,y)$ to represent "x is the mother of y." The original statement can be represented as

$$\forall x((F(x) \wedge P(x) \rightarrow \exists y\, M(x,y).$$

Using the null quantification rule we can move $\exists y$ to the left so that it appears just after $\forall x$, because y does not appear in $F(x) \wedge P(x)$. We obtain the logically equivalent expression

$$\forall x\, \exists y\, ((F(x) \wedge P(x) \rightarrow M(x,y).$$

Example 4. Let $L(x,y)$ be the statement "x loves y," where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.
(a) Everybody loves Raymond.

$$\forall x\, L(x, Raymond).$$

(b) Everybody loves somebody.

$$\forall x\, \exists y\, L(x,y).$$

(c) There is somebody whom everybody loves.

$$\exists y\, \forall x\, L(x,y).$$

(d) Nobody loves everybody.

$$\forall x\, \exists y\, \neg L(x,y) \qquad \text{or} \qquad \neg\exists x\, \forall y\, L(x,y).$$

(e) Everyone loves himself or herself

$$\forall x\, L(x,x).$$

Example 5. Express the statement "Everyone has exactly one best friend" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.
Ans: The statement "Everyone has exactly one best friend" can be expressed as "For every person x, person x has exactly one best friend." Introducing the universal quantifier, we see that this statement is the same as "$\forall x$ (person x has exactly one best friend)," where the domain consists of all people. To say that x has exactly one best friend means that there is a person y who is the best friend of x, and furthermore, that for every person z, if person z is not person y, then z is not the best friend of x. When we introduce the predicate $B(x,y)$ to be the statement "y is the best friend of x," the statement that x has exactly one best friend can be represented as

$$\exists y(B(x,y) \wedge \forall z((B(x,z) \rightarrow z = y) \qquad \text{or} \qquad \exists y(B(x,y) \wedge \forall z((z \neq y) \rightarrow \neg B(x,z))$$

and our statement is written as

$$\forall x\, \exists y(B(x,y) \wedge \forall z((B(x,z) \rightarrow z = y)).$$

1.1 Negating nested quantifiers

We can use De Morgan's laws to move the negation inside the quantifiers.

Example 6. Express the negation of the statement $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier.
Ans: We want to express $\neg\forall x \exists y(xy = 1)$ in a different way, moving the negation pass the quantifiers:

$$\neg\forall x \exists y(xy = 1) \equiv \exists x \neg(\exists y(xy = 1)) \equiv \exists x \forall y \neg(xy = 1).$$

This last one can be expressed as $\exists x \forall y(xy \neq 1)$.

Example 7. Use quantifiers to express the definition of the limit of a real-valued function $f(x)$ of a real variable x at a point a in its domain.
Ans: We say that the limit of $f(x)$ as x approaches a is L when: For every real number $\epsilon > 0$ there exists a real number δ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. This definition of a limit can be phrased in terms of quantifiers by

$$\forall \epsilon > 0 \exists \delta > 0 \forall x(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon).$$

where the domain for the variables are all real numbers.

Example 8. Now we are going to use quantifiers and predicates to express the fact that $\lim_{x\to a} f(x)$ **does not exist**, where $f(x)$ is a real-valued function of a real variable x and a belongs to the domain of f.
Ans: To say that the limit $\lim_{x\to a} f(x)$ does not exist means that for all real numbers L, $\lim_{x\to a} f(x) \neq L$. The statement $\lim_{x\to a} f(x) \neq L$ can be expressed as

$$\neg(\forall \epsilon > 0 \exists \delta > 0(\forall x(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)).$$

We can see that $\neg(\forall \epsilon > 0 \exists \delta > 0(\forall x(0 < |x-a| < \delta \rightarrow |f(x)-L| < \epsilon)))$ is equivalent to $\exists \epsilon > 0 \neg(\exists \delta > 0(\forall x(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)))$ and this last one is equivalent to

$$\exists \epsilon > 0 \forall \delta > 0 \;\; \neg(\forall x(0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)))$$

or

$$\exists \epsilon > 0 \forall \delta > 0 \exists x \neg \; (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon).$$

This last one is equivalent to

$$\exists \epsilon > 0 \forall \delta > 0 \exists x \, (0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon).$$

Questions:

(1) Let $F(x, y)$ be the statement "x can fool y," where the domain consists of all people in the world. Use quantifiers to express each of these statements.
(a) Everybody can fool Fred.
(b) Evelyn can fool everybody.
(c) Everybody can fool somebody.
(d) There is no one who can fool everybody.
(e) Everyone can be fooled by somebody.
(f) No one can fool both Fred and Jerry.
(g) Nancy can fool exactly two people.